Optimum Grasp Has Equal Friction Angles

A. Subramanian, S. Mukherjee

Department of Mechanical Engineering,
IIT Delhi
N. Delhi 110016

ABSTRACT

The three-point contact problem necessitates use frictional contacts as opposed to non-frictional contacts to oppose all possible wrenches. Use of three frictional contacts however leads to an underspecified system, introducing the possibility of optimisation. A grasp minimising the possibility of slip is obtained on minimising the maximum friction angle at each of the three contact points. An elegant solution to this minimax problem is achieved by decomposing the force system into an equilibrating force field and an interaction force field. Using this approach, it is proven that optimum grasp is achieved when at least two of the friction angles are equal. For the example of a cube loaded under its own weight, we show numerically, that optimum grasp is achieved when the three friction angles are equal.

1. INTRODUCTION

The grasp system shown in figure 1 above has three point contact forces \( P_1, P_2, \) and \( P_3. \) Governing equations for an object in motion can be posed as a static equilibrium problem by using D’Alembert’s principle. Let \( \mathbf{F} \) and \( \mathbf{M} \) be the external force and moment
loading the object. Then equilibrium conditions for a grasped object may then be represented in the form of a single function \( g \) such that,

\[
\begin{bmatrix}
F \\
M
\end{bmatrix} = g \begin{bmatrix}
F_{1x} \\
F_{2x} \\
\vdots \\
F_{nx}
\end{bmatrix}
\tag{1}
\]

Alternatively this functional relationship may be expressed by equation 2 below:

\[
wGq + w = 0
\tag{2}
\]

It is not physically possible for the finger to apply a force whose inner product with the inward drawn normal at the point of contact is negative or to exceed the friction limits. These constraints have to be incorporated into the solution procedure. The contact forces have to be recomputed for successive time steps even for the same set of contact points because of the changing loads seen by the system. However satisfaction of necessary conditions such as positive contact forces cannot be guaranteed by the minimum norm solution alone. In order to satisfy these conditions null solutions have to be superimposed upon the minimum norm solution.

2. EQUILIBRATING AND INTERACTION FORCES

A system of contact forces can be decomposed into an equilibrating force field and an interaction force field. The interaction force between two points of contact is the component of the difference of the contact forces along the line joining the two contact points. This can be mathematically expressed as follows:

\[
F_{\text{int}} = \frac{(P_i - P_j) (R_i - R_j)}{|R_i - R_j|}
\tag{3}
\]

where \( F_{\text{int}} \) is the interaction force, \( P_i \) and \( P_j \) are the contact forces at points \( i \) and \( j \) respectively and \( R_i \) and \( R_j \) are the respective position vectors for points \( i \) and \( j \).
The equilibrating forces are the forces required for equilibrium against an external load. These forces have no interaction force components. If \( \mathbf{F}_i \) and \( \mathbf{F}_j \) are equilibrating force components at two contact points \( i \) and \( j \), we must have:

\[
(\mathbf{F}_i - \mathbf{F}_j) \cdot (\mathbf{R}_j - \mathbf{R}_i) = 0
\]  

(4)

Kumar and Waldron (1988) show that if, among other conditions, a body is subjected to multiple frictional contacts then the Moore-Penrose generalised inverse solution to the equilibrium equation yields a solution vector which lies completely in the equilibrating force field and has no interaction force component. They present an efficient algorithm to compute the equilibrating forces (the minimum norm solution) for specified points of contact and loading conditions. Their method does not require one to go through the pseudo inverse calculation.

The minimum norm solution is the set of equilibrating forces. The null solution is the set of interaction forces. Superimposition of a linear combination of null solutions upon the minimum norm solution can be used to modify the contact conditions.

If \( \mathbf{T} \) and \( \mathbf{Q} \) are the net external torque and force acting on the system and \( \mathbf{I} \) is the centroidal moment of inertia given by:

\[
\mathbf{I} = \begin{bmatrix}
\bar{y}^2 + \bar{z}^2 & -\bar{x}\bar{y} & -\bar{x}\bar{z} \\
-\bar{x}\bar{y} & \bar{z}^2 + \bar{x}^2 & -\bar{y}\bar{z} \\
-\bar{x}\bar{z} & -\bar{y}\bar{z} & \bar{y}^2 + \bar{x}^2
\end{bmatrix}
\]

(5)

where,

\[
\bar{x}^2 = \frac{1}{n} \sum_{i=1}^{n} (x_i - \bar{x})^2 \\
\bar{y}^2 = \frac{1}{n} \sum_{i=1}^{n} (y_i - \bar{y})^2 \\
\bar{z}^2 = \frac{1}{n} \sum_{i=1}^{n} (z_i - \bar{z})^2
\]

\[
\bar{y} = \frac{1}{n} \sum_{i=1}^{n} (y_i - \bar{y})(z_i - \bar{z}) \\
\bar{x} = \frac{1}{n} \sum_{i=1}^{n} (x_i - \bar{x})(z_i - \bar{z}) \\
\bar{z} = \frac{1}{n} \sum_{i=1}^{n} (z_i - \bar{z})(x_i - \bar{x})
\]

with screw magnitude \( L \), screw axis \( \mathbf{u} \), screw pitch \( h \), and reference vector \( \rho_n \).
given by \( L = \frac{1^{-1}T}{n} \); \( \mathbf{u} = \frac{1^{-1}T}{nL} \); \( h = \frac{1}{nL}(\mathbf{u} \cdot \mathbf{Q}) \); \( \rho_n = \frac{1}{nL}(\mathbf{u} \times \mathbf{Q}) \), the force that
the equilibrating force must counter at point \( i \) is,
\[
\mathbf{F}_i = L \mathbf{u} \times (\mathbf{r}_i - \rho_n) + hL \mathbf{u}
\]  
(6)

All terms in equation 6 are defined except for \( \mathbf{r}_i \) which is the position vector of point \( i \) with respect to the origin which lies at the centroid of the polygon formed by the \( n \) points of contact. For the three-point problem, \( n = 3 \), and the origin lies at the centroid of the triangle formed by the three contact points. Superimposing the null solutions to formulate an expression for the net force at a contact point:
\[
\mathbf{P}_1 = \mathbf{F}_1 + k_{12} \mathbf{u}_{12} + k_{13} \mathbf{u}_{13}
\]
\[
\mathbf{P}_2 = \mathbf{F}_2 + k_{23} \mathbf{u}_{23} + k_{21} \mathbf{u}_{21}
\]
\[
\mathbf{P}_3 = \mathbf{F}_3 + k_{31} \mathbf{u}_{31} + k_{32} \mathbf{u}_{32}
\]
(7)
where \( k_{ij} = k_{ji} \) and \( \mathbf{u}_{ij} = (\mathbf{r}_i - \mathbf{r}_j)/|\mathbf{r}_i - \mathbf{r}_j| \) making \( \mathbf{u}_{ij} = - \mathbf{u}_{ji} \). In this process we have added equal and opposite forces along the lines of contact between successive points (the null solution). The scalar \( k_{ij} \) represents the interaction force (between points \( i \) and \( j \)) in magnitude. It is this factor that is unknown and must be varied in order to yield an optimum solution. \( \mathbf{P}_i \) are the net forces and \( \mathbf{F}_i \) the equilibrating forces at contact point \( i \).

3. **OPTIMUM GRASP**

Correspondingly the friction angle \( \theta_i \) at point \( i \) is obtained from the relationship,
\[
\cos \theta_i = \frac{\mathbf{n}_i \cdot \mathbf{P}_i}{\| \mathbf{P}_i \|}
\]  
(8)
where \( \mathbf{n}_i \) is the unit (inward) normal to the surface at contact point \( i \).

The limiting friction problem can be formulated as that of finding a set of internal forces such that the angle \( \theta_i \) does not exceed the maximum allowable friction angle. A grasp condition, which satisfies this condition, is a **stable grasp**. This is achieved by
minimizing the maximum of the three friction angles for a set of three contact points.

The grasp plane is the plane containing the three points of contact. It is assumed that the inward drawn contact normals and the equilibrating forces are on the same side of the grasp plane. If this condition is violated, zero friction-angle cannot be achieved by manipulating the interaction force field. The optimum grasp problem can be stated as: *Given a set of contact points, the set of surface normals at the point of contact, and the force system loading the object optimize the set of forces which satisfies the equilibrium equation and minimizes the maximum friction angle at the points of contact.*

Mukherjee and Waldron [1992] analysed the case of maintaining stability while manipulating the objects and have proven that optimum grasp is achieved when at least two of the friction angles are equal. Ji and Roth [3] have shown that if the loading is in the plane of the grasp, then the optimum solution is when the three friction angles are equal. By using the example of a cube loaded under its own weight, we show numerically, that optimum grasp is achieved when the three friction angles are equal.

### 4. NUMERICAL OPTIMIZATION

We took the example of a cube loaded under its own weight with its body diagonal coincident with the z-axis. No moment appears due to the weight of the body because of symmetry. The mathematical manipulations involved in achieving such an orientation require the use of the rotation matrix expression in Eq. 9.

\[
^d R_B = \begin{bmatrix}
x x V \phi + C \phi & x y V \phi - r_z S \phi & x z V \phi + r_y S \phi \\
x y V \phi + r_z S \phi & y y V \phi + C \phi & y z V \phi - r_x S \phi \\
x z V \phi - r_y S \phi & y z V \phi + r_x S \phi & z z V \phi + C \phi 
\end{bmatrix}
\]  

(9)

where \( \phi \) is the angle that the cube has to be rotated by to align its body diagonal with the z-axis. The term \( V \phi = \text{vers} \phi = 1 - \cos \phi \). \( C \phi \) and \( S \phi \) are the cosine and sine of \( \phi \) respectively. In terms of the object frame \( \Sigma_B \), the cube has its z-axis coincident with one of the edges of the cube. From considerations of geometry, the angle \( \phi \) can thus be found to be \( \cos^{-1}(0.5773) = 54.73^\circ \). The terms \( r_x, r_y, \) and \( r_z \) are the direction cosines of the axis.
of rotation. In this case, they are respectively 0.707, −0.707, and 0.

![Diagram of rotation](attachment:image.png)

Axis of rotation: 0.707 \( \mathbf{i} \) - 0.707 \( \mathbf{j} \)

**Figure 2.** (a) Cube with an edge coincident with the z-axis; (b) Cube after rotation with body diagonal coincident with the z-axis.

The cube in this condition is symmetrical in all respects. For simplicity of calculation, center-points of 3 mutually perpendicular cube faces are chosen as the points of contact. The weight vector for all cases is assumed to be directed along the global \( z \)-axis.

By considerations of symmetry the moment due to the weight vector is zero. It is possible to analytically calculate the optimum interaction forces for this case. Since symmetry suggests that all the \( k_{ij} \)'s should be equal, and balancing the force configuration we get for a cube weight of \( w = 5 \), we get \( k = 1.3608 \) for optimum friction angle.

As a proof of convergence of the numerical procedures, it is verified that the optimization algorithm, implemented on MATLAB, yields the same value of \( k \) as calculated above. Subsequently the optimization routine was run on asymmetrical cases...
for which no clean analytical solutions exist. We “moved” the cube from the $z$-axis and about the $0.707 \mathbf{i} - 0.707 \mathbf{j}$ axis from $-15^\circ$ to $+15^\circ$, in intervals of $1^\circ$ (Fig. 3).

![Diagram of a cube in motion from an initial angle of $\alpha = -15^\circ$ to a final angle of $\alpha = 15^\circ$. The arced arrow shows the direction of motion.]

**Figure 3.** Cube in motion from an initial angle of $\alpha = -15^\circ$ to a final angle of $\alpha = 15^\circ$. The arced arrow shows the direction of motion.

The interaction forces and friction angles are recomputed for each of the three points at every $1^\circ$ interval and plotted (Fig. 4). We let the angle of rotation be represented by $\alpha$. When the body diagonal is coincident with the $z$-axis $\alpha$ equals zero. It can be seen that the values of $\theta$ follow each other closely, suggesting that in general, like the case of the planar grasp, the optimum values would occur when the friction angles are equal.

Further for rotations up to $15^\circ$, the friction angle can be controlled to deviate from the normal by only about $10^\circ$ for the loading considered.
5. RESULTS AND CONCLUSIONS

Numerical results suggest that the optimum friction angles are achieved when the individual angles of contact are equal. This result is of significance as the equality condition introduces two additional constraints and thus reduces the optimization space to that of a single variable from the set of $k_{ij}$. This leads to the possibility of evolving a polynomial equation in a single variable, which can be solved efficiently.

6. REFERENCES


Figure 4. Variation of $k_{ij}$ and the 3 angles of friction, $\theta_1$, against the range of values of $\alpha$. 