Vibration Analysis of Coupled Structures using Impedance Coupling Approach

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Abstract

The method of impedance coupling has been applied to predict frequency response functions (FRFs) for an assembly of beam structures. The influence of coordinate incompleteness and noise in the FRF data on the accuracy of the predicated FRFs is studied via simulated experiments. A method based on finite difference is used to determine FRFs that are normally unavailable by measurement. The results obtained in these studies indicate that the finite difference based determination of unavailable FRFs has the potential to be a good solution to tackle the problem of coordinate incompleteness in coupling analysis.

Introduction

There are many instances in which it is convenient to be able to consider a complex engineering structure as an assembly of simpler components or substructures. This is the general approach in which a complex structure could be regarded as being formed of different substructures (or components), each of which could first be analyzed individually and independently from the others. In this way, before being assembled to form the complete structure, each analysis could be done by whichever method was most convenient and eventually the substructure models could be assembled together to obtain the equations related to the complete structure. This is the basic idea behind the coupling approach of dynamic analysis of structures. Thus, in principle the coupling approach can entertain data from different sources that is analytical or experimental studies.

The coupling techniques are mainly classified in two categories- the Impedance Coupling approach and the Modal Coupling approach. The Impedance Coupling approach, also known as FRF Coupling approach, uses frequency response function (FRF) data of the substructures in order to synthesize the FRFs of the coupled or the assembled structure. On the other hand the Modal Coupling approach
uses modal data in order to obtain the modal data of the coupled structure.

FRF-coupling is regarded to have started with the work of Bishop and Johnson [1]. A way was formulated for calculating the dynamic predictions of multi-beam assemblies from an “exact” formulation of the response model of each individual beam based on theoretically derived models. The standard details of the method can be found in Ewins [2]. In this paper some of the important issues that affect the performance of the Impedance coupling method are examined. The influence of coordinate incompleteness and noise in the FRFs on the quality of the predicted FRFs is investigated through a simulated example of coupling of beam structures. A finite difference based method is proposed to address the problem of lack of coordinates that would be available by measurement.

Theory

In this section the theory of Impedance coupling approach to predict (FRFs) of the coupled structure on the basis of FRFs of the substructures is described. A method based on finite difference is proposed in order to obtain an estimate for the rotational degree of freedom FRFs (RDOF-FRFs).

**Impedance coupling approach:** Let A and B are two substructures that are to be coupled, to obtain coupled structure C, through DOFs j on each of them. Subscript ‘i’ represents the internal DOFs on substructure A and B. After coupling, displacement and forces at the coupling/joint DOFs (denoted by subscript ‘j’) must satisfy following conditions-

\[
\begin{align*}
\{x_j\}_A &= \{x_j\}_B = \{x_j\}_C \quad (1) \\
\{f_j\}_C &= \{f_j\}_A + \{f_j\}_B \quad (2)
\end{align*}
\]

For the substructure-A separated from the coupled structure C the displacement vector for joint DOFs can be written as,

\[
\{x_j\}_A = \begin{bmatrix} H_{ji} \end{bmatrix}_A \{f_i\}_A = \begin{bmatrix} H_{jj} & H_{ji} \end{bmatrix}_A \begin{bmatrix} f_j \\ f_i \end{bmatrix}_A = \begin{bmatrix} H_{jj} \end{bmatrix}_A \{f_j\}_A + \begin{bmatrix} H_{ji} \end{bmatrix}_A \{f_i\}_A \quad (3)
\]

Similarly for substructure B following equation can be written,

\[
\{x_j\}_B = \begin{bmatrix} H_{ji} \end{bmatrix}_B \{f_i\}_B = \begin{bmatrix} H_{jj} & H_{ji} \end{bmatrix}_B \begin{bmatrix} f_j \\ f_i \end{bmatrix}_B = \begin{bmatrix} H_{jj} \end{bmatrix}_B \{f_j\}_B + \begin{bmatrix} H_{ji} \end{bmatrix}_B \{f_i\}_B \quad (4)
\]

Equating (3) and (4) in the light of (1) and making substitution for \{f_j\}_B from (2) following equation is obtained,

\[
\{f_j\}_A = \begin{bmatrix} H_{ji} \end{bmatrix}_A + \begin{bmatrix} H_{ji} \end{bmatrix}_B \begin{bmatrix} H_{jj} \\ H_{ji} \end{bmatrix}_B \{f_j\}_C + \begin{bmatrix} H_{ji} \end{bmatrix}_B \{f_i\}_B - \begin{bmatrix} H_{ji} \end{bmatrix}_A \{f_i\}_A \quad (5)
\]
On substituting \( \{f_j\}_A \) from (5) in (3) and simplifying we get,

\[
\{x_j\}_A = \left( [H_{ji}]_A - [H_{ji}]_A ([H_{ji}]_A + [H_{ji}]_B)^{-1} [H_{ji}]_A \right) \{f_j\}_A + [H_{ji}]_A ([H_{ji}]_A + [H_{ji}]_B)^{-1} [H_{ji}]_B \{f_j\}_B
\]

But with respect to coupled structure C, \( \{x_j\}_A \) can be written as,

\[
\{x_j\}_A = [H_{ji}]_{C,CA} \{f_j\}_A + [H_{ji}]_{C,C} \{f_j\}_C + [H_{ji}]_{C,CB} \{f_j\}_B
\]

Subscripts C, CA, CB denote reference to joint/couple DOFs, internal DOFs of coupled structure that were part of substructure A and internal DOFs of coupled structure that were part of substructure B respectively. Comparing equation (6) and (7) the FRFs for the coupled DOFs of the coupled structure with respect to excitation at joint and internal DOFs associated with C, CA and CB are given by the following relationships. The FRFs appearing on the RHS of following equation are for the substructures A and B.

\[
\begin{align*}
[H_{ji}]_{C,CA} &= [H_{ji}]_A - [H_{ji}]_A ([H_{ji}]_A + [H_{ji}]_B)^{-1} [H_{ji}]_A \\
[H_{ji}]_{C,C} &= [H_{ji}]_A ([H_{ji}]_A + [H_{ji}]_B)^{-1} [H_{ji}]_B \\
[H_{ji}]_{C,CB} &= [H_{ji}]_A ([H_{ji}]_A + [H_{ji}]_B)^{-1} [H_{ji}]_B
\end{align*}
\]

By proceeding in a similar manner the other FRFs for the coupled structure can be derived. FRFs for the internal DOFs of the coupled structure associated with CA are given by following relationships.

\[
\begin{align*}
[H_{ii}]_{CA,CA} &= [H_{ii}]_A - [H_{ij}]_A ([H_{ij}]_A + [H_{ij}]_B)^{-1} [H_{ij}]_A \\
[H_{ij}]_{CA,C} &= [H_{ij}]_A ([H_{ij}]_A + [H_{ij}]_B)^{-1} [H_{ij}]_B \\
[H_{ii}]_{CA,CB} &= [H_{ij}]_A ([H_{ij}]_A + [H_{ij}]_B)^{-1} [H_{ij}]_B
\end{align*}
\]

FRFs for the internal DOFs of the coupled structure associated with CB are given by following relationships.

\[
\begin{align*}
[H_{ii}]_{CB,CA} &= [H_{ij}]_B ([H_{ij}]_A + [H_{ij}]_B)^{-1} [H_{ij}]_A \\
[H_{ij}]_{CB,C} &= [H_{ij}]_B - [H_{ij}]_B ([H_{ij}]_A + [H_{ij}]_B)^{-1} [H_{ij}]_B \\
[H_{ii}]_{CB,CB} &= [H_{ij}]_B - [H_{ij}]_B ([H_{ij}]_A + [H_{ij}]_B)^{-1} [H_{ij}]_B
\end{align*}
\]
Simulated Study

In this section a simulated case study of assembly of two cantilevers, as shown in Figure (1), to obtain a fixed-fixed beam as the coupled structure is considered. The dimensions of the two cantilevers are 1000×50×5 mm. The modulus of elasticity and density are taken as 2.0e+11 N/m² and 7800 kg/m³ respectively. Each cantilever is modeled using five beam elements with the node at one end fixed giving a total of six nodes with three degrees of freedom (deflection due to bending, axial deformation and rotation/slope) each. Figure (2) shows joint-DOFs 16(axial), 17(translation) and 18(rotational) for substructure A and joint-DOFs 1(axial), 2 (translation) and 3 (rotational) for substructure B. Figure (2) also shows translation-DOFs 8 and 11 as internal DOFs on substructure A for which FRF \([H_{8,11}]_{CA,CA}\) corresponding to the coupled structure is to be predicted. On the basis of equation (9) it can be seen that the following FRFs should be available via measurement to predict \( [H_{8,11}]_{CA,CA} \) exactly:

\[
\begin{align*}
\text{a) Translational FRFs:} & \quad \begin{bmatrix} H_{8,11} \end{bmatrix}_A, \begin{bmatrix} H_{8,17} \end{bmatrix}_A, \begin{bmatrix} H_{11,17} \end{bmatrix}_A, \begin{bmatrix} H_{17,17} \end{bmatrix}_A, \begin{bmatrix} H_{2,2} \end{bmatrix}_B \\
\text{b) Rotational FRFs:} & \quad \begin{bmatrix} H_{18,8} \end{bmatrix}_A, \begin{bmatrix} H_{18,11} \end{bmatrix}_A, \begin{bmatrix} H_{18,17} \end{bmatrix}_A, \begin{bmatrix} H_{18,18} \end{bmatrix}_A, \begin{bmatrix} H_{3,3} \end{bmatrix}_B, \begin{bmatrix} H_{3,2} \end{bmatrix}_B \\
\text{c) FRFs related to Axial – DOF at the Joint:} & \quad \begin{bmatrix} H_{11,16} \end{bmatrix}_A, \begin{bmatrix} H_{16,16} \end{bmatrix}_A, \begin{bmatrix} H_{3,3} \end{bmatrix}_B, \begin{bmatrix} H_{2,2} \end{bmatrix}_B, \begin{bmatrix} H_{1,1} \end{bmatrix}_B 
\end{align*}
\]

**Complete data:** Case of Complete data means that all the FRFs, Translation as well as Rotational and axial FRFs mentioned in (11) are assumed to be available by measurement. The FRF \([H_{8,11}]_{CA,CA}\) predicted by impedance coupling of substructure FRFs using equation (9) is shown in Figure (3). The predicted FRF has been overlaid on the corresponding exact FRF that is obtained on the basis of the direct assembly of structural matrices of substructures. Due to completeness of the data the predicted FRF is exactly identical to the exact FRF.

**Incomplete data:** In practice, no reliable and accurate methods are available for the measurement of RDOF-FRFs and therefore these FRFs are not generally available as do translation FRFs. In the case of incomplete data the influence of unavailability of some of the FRFs on the accuracy of predicted FRFs for the coupled structure is studied. First it is assumed that none of the FRFs related to the axial joint DOFs, 16 on substructure A and 1 on substructure B and as mentioned in equation (11), are available. The FRF \([H_{8,11}]_{CA,CA}\) predicted by using equation (9) is shown in Figure (4). It is seen that the predicted FRF for the coupled structure has a very good fit with the corresponding exact FRF. This must be because of the fact that the DOFs, translation DOFs 8 and 11, for which the FRF has been predicted are not coupled to axial DOF. Therefore, whether the coupling of the two substructures through the axial DOF exists or not does not affect the prediction accuracy.
Next, it is assume that all the FRFs related to axial DOF as well as rotational DOF are not available and only the remaining FRFs related to translation DOFs are available. The FRF $[H_{8,11}]_{CA,CA}$ predicted for this case is shown in Figure (5). It is seen that the predicted FRF for the coupled structure is deviating significantly from the exact FRF. In this case of incomplete data the two substructures are effectively coupled only via the translation DOF, as RDOF-FRF is not available. Consequently, there is no provision for the exchange of moment between the two substructures that have been coupled. Thus, it is concluded that information about RDOF-FRFs is very essential if any meaningful prediction for the FRFs of the coupled structure is to be made.

**Determination of RDOF-FRFs:** In this work a method is proposed to obtain an estimate for the substructure-RDOF-FRFs that are required for making prediction of FRFs for the coupled structure. The method is based on finite difference scheme applied to translation FRFs that can be measured easily. To estimate an RDOF-FRF associated with a point the translation FRF at that point and a nearby point situated at some distance $\epsilon$ are utilized (see Figure (2)). The formulas for estimating RDOF-FRFs, required as per equation (11), using information about translation FRFs are developed below.

\[
[H_{18,17}]_A = \frac{[H_{17,17}]_A - [H_{a,17}]_A}{\epsilon} \quad [H_{18,8}]_A = \frac{[H_{17,8}]_A - [H_{a,8}]_A}{\epsilon}
\]

\[
[H_{18,11}]_A = \frac{[H_{17,11}]_A - [H_{a,11}]_A}{\epsilon} \quad [H_{3,2}]_B = \frac{[H_{b,2}]_B - [H_{2,2}]_B}{\epsilon}
\]

\[
[H_{18,18}]_A = \frac{[H_{17,18}]_A - [H_{a,18}]_A}{\epsilon} = \frac{[H_{18,17}]_A - [H_{18,a}]_A}{\epsilon}
\]

\[
= \frac{1}{\epsilon} \left( [H_{17,17}]_A - [H_{a,17}]_A - \frac{[H_{17,a}]_A - [H_{a,a}]_A}{\epsilon} \right)
\]

\[
= \frac{[H_{17,17}]_A - 2[H_{a,17}]_A + [H_{a,a}]_A}{\epsilon^2}
\]

\[
[H_{3,3}]_B = \frac{[H_{2,2}]_B - 2[H_{2,b}]_B + [H_{b,b}]_B}{\epsilon^2}
\]

It is noted that the use of equation (12) to obtain an estimate for an RDOF-FRF would require measurement of some additional translation FRFs. In the present work simulated translation FRFs for the DOFs $a$ and $b$ on substructures $A$ and $B$ respectively situated at distance of $\epsilon=2\text{cm}$ are generated additionally for use in finite difference formulas. Figures (6a) and (6b) show overlays of RDOF-FRFs $[H_{18,8}]_A$ and $[H_{18,18}]_A$ estimated using above scheme with their corresponding exact RDOF-FRF obtained on the basis of direct assembly of substructure structural matrices. The figure indicates that the predicted RDOF-FRFs are quite closely following the corresponding exact FRFs.
The above estimates for the required RDOF-FRFs are then used in equation (9) to predict the FRF \([H_{8,11}]_{CA,CA}\). The predicted FRF is found to be almost identical with the exact FRF as shown in Figure (7).

**Effect of noise:** In practice, the measured translation FRFs would inherently contain some measurement noise. To assess the effect of noise on the quality of the estimated RDOF-FRFs and the predicted FRFs, the translation FRFs that forms the input for the prediction process are polluted with a random noise of 5%. The estimated RDOF-FRFs \([H_{18,8}]_A\) and \([H_{18,18}]_A\) are shown in Figure 8(a) and 8(b) respectively while Figure (9) shows the predicted FRF \([H_{8,11}]_{CA,CA}\). It is seen that the predicted FRF is reasonably close to the exact FRF though in some regions there is more noise in the predicted FRFs.

**Effect of distance \(\varepsilon\) on the predicted FRF:** Required RDOF-FRFs are estimated for values of \(\varepsilon\) equal to 2cm, 4cm, 6cm and 8cm. Figures 10(a), 10(b), 10(c) and 10(d) indicate the estimates of RDOF-FRF \([H_{18,18}]_A\) for these values respectively. The predicted FRF \([H_{8,11}]_{CA,CA}\) for these four sets of estimates of RDOF-FRFs is shown in Figure 11(a), 11(b), 11(c) and 11(d). It is seen that even up to a distance of \(\varepsilon\) equal to 6cm the estimated RDOF-FRF is not in much error. This result indicates that even if an accelerometer, for FRF-measurement, is not placed very close to the point where RDOF-FRF is required even then there is a possibility of obtaining a reasonably good estimate for the RDOF-FRF and for the predictions based on that. This fact of course needs to be verified for the complex cases and using actual experimental data.

**Conclusion**

The issues of coordinate incompleteness and noise in the FRFs for structural coupling using impedance coupling approach are studied through a simulated study. It is concluded that in case of coupling of beam structures the availability of FRFs associated with rotational degrees of freedom is very crucial for predicting accurately the FRFs for the coupled structure. The unavailability of FRFs associated with axial degrees of freedom has a negligible effect on the prediction accuracy. The method based on finite difference gives a reasonably good estimate of RDOF-FRFs. However, some additional FRFs are required to be measured close to the point where RDOF-FRF is to be estimated by finite difference approach. It is observed in the present study that even if simulated FRFs that are used for this purpose are not very close to the point where RDOF-FRF is required still the predicted FRF are reasonably close to the exact FRF.

**References**


**Notations**
\[H\]: Frequency Response Function (FRF) matrix
\{x\}: Vector of displacements
\{f\}: Vector of forces
FRF: Frequency response function
DOF: Degrees of freedom
RDOF: Rotational DOF

**Subscripts:**
i : Internal DOFs
j: Joint DOFs
t: All/total DOFs (Internal + Joint)
CA: Those DOFs of coupled structure that were part of substructure A
CB: Those DOFs of coupled structure that were part of substructure B
C: Coupled DOFs, Coupled structure
A,B : Substructure A & B

Figure 1: Substructures to be coupled
Figure 2: DOFs involved for prediction of FRF
\[H_{8,11}]_{CA,CA}\] for the coupled structure

Figure 3: Overlay of Predicted FRF
\[H_{8,11}]_{CA,CA}\] (---) and the exact FRF (---) for the case of complete data
Figure 4: Overlay of predicted FRF
\[H_{8,11}]_{CA,CA}\] (---) and the exact FRF (---) for the case of incomplete data when none of the FRFs related to axial-DOF of joint are available
Figure 5: Overlay of predicted FRF $[H_{8,11}]_{CA,CA}$ (---) and the exact FRF (___) for the case of incomplete data when the FRFs related to only translation-DOFs are available

Figure 7: Overlay of predicted FRF $[H_{8,11}]_{CA,CA}$ (---) and the exact FRF (___) for the case of incomplete data and using estimated RDOF-FRFs

Figure 6(a) & (b): Overlay of estimates of the RDOF-FRF $[H_{18,8}]_A$ and $[H_{18,18}]_A$ (---) and the corresponding exact RDOF-FRFs (___)

Figure 8(a) & (b): Overlay of estimates of the RDOF-FRF $[H_{18,8}]_A$ and $[H_{18,18}]_A$ (---) and the corresponding exact RDOF-FRFs (___) when the simulated translation FRFs are polluted with noise
Figure 9: Overlay of predicted FRF $[H_{8,11}]_{CA,CA}$ (---) and the exact FRF (____) for the case of incomplete data and using estimated RDOF-FRFs under the presence of noise.

Figure 10 (a, b, c & d): Overlay of estimates of the RDOF-FRF $[H_{18,8}]_A$ and $[H_{18,18}]_A$ (---) and the corresponding exact RDOF-FRFs (____) for values of $\epsilon$ equal to 2cm, 4cm, 6cm and 8cm respectively for the case of incomplete data under the presence of noise.
Figure 11 (a, b, c & d): Overlay of predicted FRF \( [H_{8,11}]_{CA,CA} \) (---) and the corresponding exact FRF (____) for values of \( \varepsilon \) equal to 2cm, 4cm, 6cm and 8cm respectively for the case of incomplete data under the presence of noise.